Puzzle: 3D Nonlinear model of the vertical spring force of a trampoline - Can mathematical approximations be compensated through curve fitting ?

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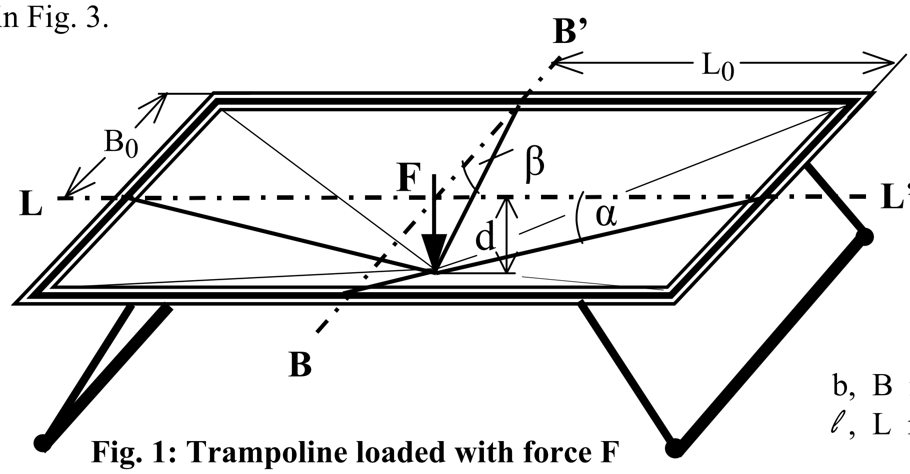
The trampoline bed of Fig. 1 is deflected at the centre through a vertical force F. The amount of vertical deflection is d. To calculate its mathematical model we use the force vector diagrams of the longitudinal-sectional view LL′ in Fig. 2 and in the cross-sectional view BB′ in Fig. 3.



b, B refer to **B**readth

ℓ, L refer to **L**ength

Fig. 1: Trampoline loaded with force F

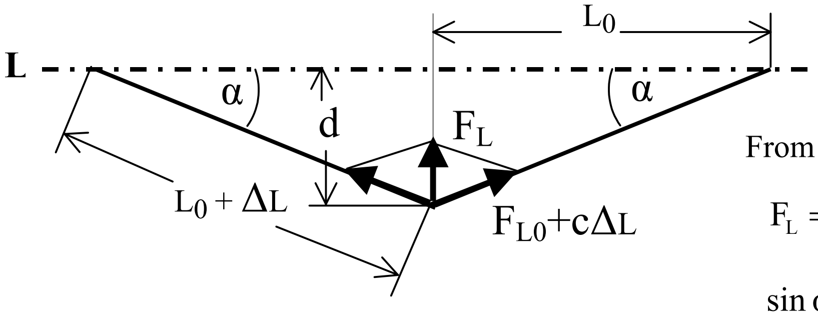


Fig. 2: Longitudinal-sectional view LL′

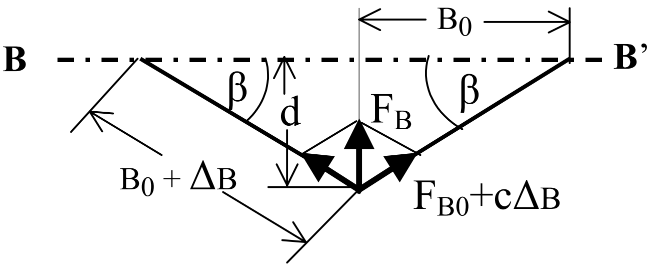


Fig. 3: Cross-sectional view BB′

From Figure 2 we have

$$F_L = 2(F_{L0} + c\Delta L)\sin\alpha \tag{1}$$

$$\sin\alpha = \frac{d}{L_0 + \Delta L} \tag{2}$$

Introduce

$$F_{L0} = c\Delta L_0 \tag{3}$$

and substitute (2) and (3) in (1) we get

$$F_L = 2cd\frac{\Delta L_0 + \Delta L}{L_0 + \Delta L} \tag{4}$$

Using Figure 3 accordingly yields

$$F_B = 2cd\frac{\Delta B_0 + \Delta B}{B_0 + \Delta B} \tag{5}$$

Add Equation (4) and (5) then

$$F = F_L + F_B \tag{6}$$

- c = effective spring constant
- L_0, B_0 = half length/breadth of the tramp
- ΔL_0, ΔB_0 = initial extension of the springs
- ΔL, ΔB = extension of the springs under load
- F_{L0}, F_{B0} = prestress force created by ΔL_0, ΔB_0
- F_L, F_B = spring force created by ΔL, ΔB
- d = deflection of the trampoline centre
- ℓ, b = half length/breadth of the rigid plate

To find an explicit solution of (6) we introduce the following approximations:

$$L_0 + \Delta L \approx L_0 \quad (7)$$

$$B_0 + \Delta B \approx B_0 \quad (8)$$

Substitute (4),(5),(7) and (8) in (6) then

$$F \approx 2cd\left[\frac{(\Delta L_0 + \Delta L)}{L_0} + \frac{(\Delta B_0 + \Delta B)}{B_0}\right] \quad (9)$$

Applying the Pythagorean theorem to either of the rectangular triangles of Fig. 2 and rearranging we obtain

$$\frac{\Delta L}{L_0} = \sqrt{1 + \frac{d^2}{L_0^2}} - 1 \quad (10)$$

Expanding the square root into a Taylor series and using the first two terms leads to

$$\frac{\Delta L}{L_0} \approx \frac{1}{2}\frac{d^2}{L_0^2} \quad (11)$$

Repeating the same procedure on either of the rectangular triangle of Fig. 3 yields

$$\frac{\Delta B}{B_0} \approx \frac{1}{2}\frac{d^2}{B_0^2} \quad (12)$$

When substituting (11) and (12) in (9) and rearranging, we have

$$F \approx 2c(\frac{\Delta L_0}{L_0} + \frac{\Delta B_0}{B_0})d + c(\frac{1}{L_0^2} + \frac{1}{B_0^2})d^3 \quad (13)$$

To further improve the models performance, a rigid weightless plate is centrally placed on the trampoline to take the foot contact area of the gymnast into consideration, which causes a force $F_1 > F$ for the same deflection d.

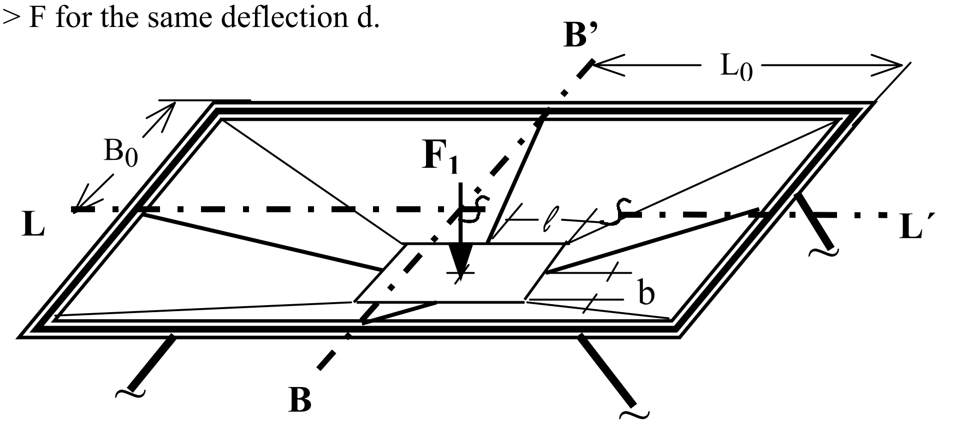


Fig. 4: Trampoline with a weightless rigid plate of an area $4b\ell$ *)

Thus the cross-sectional view has changed and is shown in Fig. 5. The modified longitudinal-sectional view can be set up accordingly when changing B to L, b to ℓ and β to α.

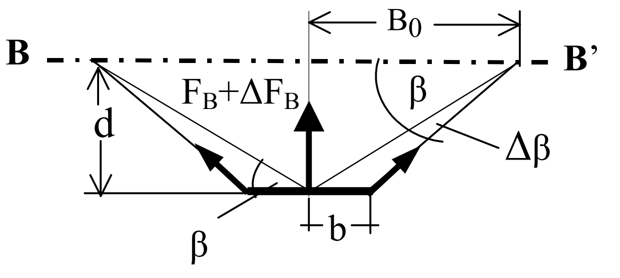


Fig. 5: Cross-sectional view of the modified model

$\Delta\beta << \beta$
ΔF_B = partial increase of the spring force due to the rigid plate
b = half breadth of the plate

and for the longitudinal view
$\Delta\alpha << \alpha$
ΔF_L = partial increase of the spring force due to the rigid plate
ℓ = half length of the rigid plate

Consequently the increase ΔF and the total spring force F_1 for the same deflection d due to the rigid plate can be calculated as

$\Delta F = \Delta F_B + \Delta F_L$ (14) and $F_1 \approx F + \Delta F$ (15) respectively.

Problems for solution

1. Use Fig. 5 and show that $$\Delta F \approx c\left(\frac{\ell}{L_0^3}+\frac{b}{B_0^3}\right)d^3 \qquad (16)$$

   Combine (13), (15) and (16) and the mathematical model of the vertical spring force F_1 of the trampoline bed with the rigid plate should come out as follows:

$$F_1 \approx 2c\left(\frac{\Delta L_0}{L_0}+\frac{\Delta B_0}{B_0}\right)d + c\left(\frac{1}{L_0^2}+\frac{1}{B_0^2}+\frac{\ell}{L_0^3}+\frac{b}{B_0^3}\right)d^3 \qquad (17)$$

2. Equation (17) is of the form $F_1 \approx k_1 d + k_3 d^3$. Check its plausibility when L_0 and B_0 approaches large values, when ℓ and b approaches zero and when ΔL_0 and ΔB_0 approaches small and large values respectively.
3. Asses the impact of the approximations due to equations (7),(8), (11), (12) and (16) on the mathematical model of equation (17).
4. The validation of the model can be made with experimental measurements at a real trampoline and through a curve fitting procedure. Can this curve fitting procedure compensate for eventual parametrical and structural errors of the model which are due to the applied mathematical approximations?
5. Write down the reasons, why the independent variable d of equation (17) only has positive odd exponents.
6. Why is it sufficient to solely use the cross-sectional and longitudinal-sectional view of Fig. 2 and Fig. 3 for setting up the mathematical model of Eq. (17) and disregard all the other possible isosceles triangles through F?

*) Kraft, M.: Eine einfache Näherung für die vertikale Federkraft des Trampolinsprungtuchs.
(A simple approximation of the vertical spring force of a trampoline bed.)
http://opus.tu-bs.de/opus/volltexte/2001/214